

Transport Channels in a Double Junction - Coherent coupling changes the picture

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Abstract. Transport through a point contact is accurately modelled by assigning to the junction an ensemble of independent transport channels with possibly different transmissions. We here argue that for a series of two contacts, coherently coupled across an island, the transport channels are different from the ensembles that would describe each contact taken as stand-alone device. We further show that instead of two sets of channels with manifold cross-links over the island the double junction can be described by pairs of channels from both sides coherently coupled together, where each pair, however, has no coherent connection to the others. This finding will substantially simplify modelling transport by a Green's functions technique. Additional channels through only one junction may complete the picture. Finally we discuss how partial coherence across the island with an appropriate ansatz can be modelled in the same scheme.

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1. Introduction

In low-dimensional electron systems [1, 2, 3] the quantization of conductance [4] is observed. For constrictions in two-dimensional electron gases the conductance adopts multiples of the conductance quantum $G_0 = e^2/h$ because the transverse wave numbers determine how many modes contribute [5, 6]. Single-atomic size contacts exhibit typical, reproducible and material dependent conductances, which, however, in general are no integer multiples of G_0 [7]. The conductance is associated to an ensemble of transport channels, the number of which corresponds to the number of valence orbitals of the element used. The individual channel transmissions in the range $0 \dots 1$ reflect the wavefunction overlap from the central to neighbouring and following atoms [8]. Although atomistic ab initio calculations greatly complement experiments in this field, they are not needed to deduce transport channels and their transmissions from measurements. The description using an ensemble of transport channels is a more general concept applicable to any sort of point contact [9]. The contact - including leads to bulk electrodes - can be viewed as a black box that behaves like the associated ensemble of transport channels. A deeper interpretation of the channels is not required. We shortly review here, in our

own way, the theoretical construction behind the description of a single contact in terms of a channel ensemble, because this basis is needed for generalizing the concept to the double junction.

2. Single junction

Regarding the left and right side of a point contact as in Fig.1a at first as uncoupled reservoirs let there be a finite number of orthonormal modes on each side (2 on the left and 3 on the right as depicted in Fig.1b, for example). These may be localized atomic orbitals or band modes of a solid crystal. ‡ (Small letters l and r are used here just to distinguish the single from the double junction.) Putting the left and right side together, scattering is determined by some complex amplitudes s_{ij} gathered into matrix S (Fig.1c). The case that some modes may not couple is included. Some entries s_{ij} may be zero. Reverse scattering is given by the adjoint matrix (Fig.1d). Not every matrix with as many lines as there are modes on one side and as many columns as there are modes on the other side can be a scattering matrix S , though. Probability conservation sets upper limits on the entries. A mandatory limit is, of course, that the absolute value of each entry be lower than or equal to 1, $|s_{ij}| \leq 1$ for all i, j . But then one could think at first glance that it would be sufficient to demand that a particle occupying a pure mode on one side after being scattered to the other side should not cause a probability exceeding 1 there, that is $\sum_i |s_{ij}|^2 \leq 1$ and then the other way round $\sum_j |s_{ji}^*|^2 \leq 1$ for every j . An example showing that this is insufficient will be given in section 4. For independent transport channels we want an input on the left to come back into the same eigenmode it came from there after having been scattered to the right and back. All further multiple reflections will then stay in this mode. The channels will therefore be determined by diagonalizing $S^\dagger S$. By construction $S^\dagger S$ is a hermitian matrix and thus has real eigenvalues. To ensure that the total probability does not increase it is required that any superposition of modes on the left described by a normalized distribution vector when scattered to the right and back is projected onto a mode distribution with total probability less than or at most 1 again on the left. The same, of course, has to hold true for starting with a normalized vector on the right and regarding the probability returning from one backreflection to the left. We shall show that the prohibition of probability creation in forth- and backreflections is equivalent to all eigenvalues of $S^\dagger S$ being less or equal to one. "All eigenvalues of $S^\dagger S \leq 1$ " is not a property to deduce of a scattering matrix S , but the definition to put for calling a matrix S a scattering matrix.

In contrast to the usual scattering formalism [3, 5] we do not distinguish incoming and outgoing modes here. Furthermore instead of two transmission and two reflection quadrants our *scattering matrix* S only consists of one transmission block and only

‡ The model later will, however, assume that each mode has a constant density of states around the Fermi energy in a range a few times the equivalent of the voltage applied over the contact or the corresponding bulk superconductor quasiparticle density of states according, for example, to BCS theory.

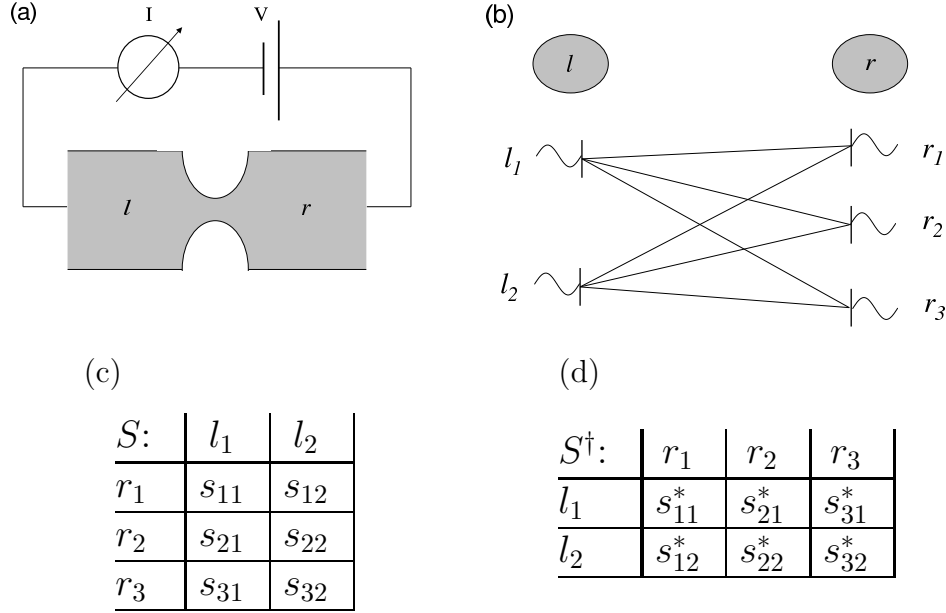


Figure 1. (a) Point contact embedded in an electric circuit. (b) Modes left and right that can arbitrarily couple. (c) Scattering matrix from left to right. (d) Scattering matrix from right to left.

describes transmission from one side of the contact to the other, and therefore does not have to be square. The term *modes* or *original modes* is used for states that conveniently describe the left and right side of the contact, which can be taken from the situation before a contact is established. We avoid the word "channel" for these, which is often used synonymous with incoming and outgoing modes [3, 5] because it is implied differently in the experiment-related description of atomic contacts [7]. We call *eigenmodes* or *new modes* those linear combinations of original modes that constitute eigenvectors of $S^\dagger S$ or SS^\dagger . Eigenmodes on the left and the right side of a contact are associated to each other one to one and their connections are called *transport channels* or simply *channels*.

Let a_{ij} denote the entries of the matrix $(\mathfrak{h}_{S^\dagger S})$ such that each column of $(\mathfrak{h}_{S^\dagger S})$ is a normalized eigenvector of $S^\dagger S$. Eigenvectors associated to eigenvalues zero have to be kept, and in case of degenerate eigenvalues choose a set of orthogonal eigenvectors associated to them. $(\mathfrak{h}_{S^\dagger S})$ is a square matrix with the dimension the number of modes on the left. The orthonormality of the eigenvectors is expressed through

$$\sum_j a_{ji}^* a_{jk} = \delta_{ik}. \quad (1)$$

Any normalized input state on the left given by a distribution vector \vec{b} in the basis of the original modes can be converted into the basis of eigenvectors, where we shall call this same vector \vec{c} . $(\mathfrak{h}_{S^\dagger S})$ is invertable. $\vec{b} = (\mathfrak{h}_{S^\dagger S})\vec{c}$ or $b_m = \sum_j a_{mj} c_j$. From $\sum_m b_m^* b_m = 1$ it easily follows that also $\sum_j c_j^* c_j = 1$ and vice versa. The following calculation shows the

above claimed equivalence about S being a scattering matrix:

$$\begin{aligned}
 |S^\dagger S \vec{b}|^2 &= \sum_i \left| \sum_j (S^\dagger S)_{ij} b_j \right|^2 \leq 1 \\
 \iff \sum_{i,j,k} (S^\dagger S)_{ij}^* b_j^* (S^\dagger S)_{ik} b_k &\leq 1 \\
 \iff \sum_{i,j,k,l,m} c_l^* (S^\dagger S)_{ij}^* a_{jl}^* (S^\dagger S)_{ik} a_{km} c_m &\leq 1 \\
 \iff \sum_{i,l,m} c_l^* (\lambda_l a_{il})^* \lambda_m a_{im} c_m &\leq 1 \\
 \iff \sum_m c_m^* \lambda_m^2 c_m &\leq 1
 \end{aligned} \tag{2}$$

$\sum_k (S^\dagger S)_{ik} a_{km} = \lambda_m a_{im}$ expresses that the m th column of $(\mathbf{q}_{S^\dagger S})$ is the eigenvector corresponding to eigenvalue λ_m . Of course, $\lambda_m^* = \lambda_m$, because eigenvalues are real, and (1) has been used.

Some normalized vector \vec{b} or equivalently any normalized vector \vec{c} can arbitrarily be chosen. \vec{c} could especially be a vector with any one component equal to 1 and all other components 0. Then obviously $\lambda_m^2 \leq 1 \Rightarrow |\lambda_m| \leq 1$ for every m individually. To initialize the above calculation from the last line, suppose that every $|\lambda_m|$ is smaller or equal to 1. Then, for a general \vec{c} , (2) simply means weighing each term in the sum $\sum_m c_m^* c_m = 1$ by a factor $\lambda_m^2 \leq 1$, which must give a result ≤ 1 .

Having obtained the scattering amplitudes from a microscopic physical model should ensure S being a scattering matrix and all eigenvalues of $S^\dagger S$ less or equal to 1. However, and also because of the need to make up number examples here, an easier criterion than calculating all eigenvalues of $S^\dagger S$ would be helpful.

$$\sum_{i,j} |(S^\dagger S)_{ij}|^2 \leq 1 \tag{3}$$

is a sufficient condition, although not a necessary one.

As a further important aspect one may wonder whether defining transport channels as (eigen-)modes from the right being backscattered only into themselves would have made a difference from having required this property for (eigen-)modes on the left. This would mean looking for the eigenvalues and eigenvectors of SS^\dagger . As SS^\dagger may have a different dimension from $S^\dagger S$, one of these matrices may have more eigenvalues than the other one. The conjecture, that one might get a greater number of transport channels as well as channels with different transmissions with an ansatz looking at backscattering from one side instead of from the other, however, is wrong. We shall now demonstrate that, if SS^\dagger is of greater dimension than $S^\dagger S$, SS^\dagger has all eigenvalues that $S^\dagger S$ has and all its remaining additional eigenvalues are zero. Note again that columns of $(\mathbf{q}_{S^\dagger S})$ are the eigenvectors of $S^\dagger S$:

$$(S^\dagger S)^{n_l} (\mathbf{q}_{S^\dagger S})^{n_l} = (\mathbf{q}_{S^\dagger S})^{n_l} (\lambda_{S^\dagger S})^{n_l} \tag{4}$$

$(\lambda_{S^\dagger S})$ is the diagonal matrix containing the eigenvalues of $S^\dagger S$ in the order of the columns of $(\mathbf{q}_{S^\dagger S})$. To get each column vector of $(\mathbf{q}_{S^\dagger S})$ multiplied by the respective λ_m , $(\lambda_{S^\dagger S})$ has to be multiplied to $(\mathbf{q}_{S^\dagger S})$ from the right. $S^\dagger S$, $(\mathbf{q}_{S^\dagger S})$ and $(\lambda_{S^\dagger S})$ are all square

matrices with dimension the number of modes n_l on the left side. Now multiply (4) by S from the left and as matrix multiplication is associative, we can view that as

$$(SS^\dagger)^{n_r} S^{n_r n_l} (b_{S^\dagger S})^{n_l} = S^{n_r n_l} (b_{S^\dagger S})^{n_l} (\lambda_{S^\dagger S})^{n_l} \quad (5)$$

(5) tells us that the columns of $S(b_{S^\dagger S})$ are eigenvectors of SS^\dagger - though not necessarily normalized - with the entries of $(\lambda_{S^\dagger S})$ as associated eigenvalues. SS^\dagger is a square matrix with dimension the number of modes n_r on the right side. S as well as $S(b_{S^\dagger S})$ has n_r lines and n_l columns. Single upper indices in (4) and (5) denote the dimension of a square matrix, double upper indices the line and column number of a rectangular matrix. Every eigenvalue λ_m of $S^\dagger S$ is an eigenvalue of SS^\dagger , too. However, we have not yet specified which number of modes, n_l or n_r , is the greater. For equal mode numbers $n_l = n_r$ we have just proven that the sets of eigenvalues of $S^\dagger S$ and SS^\dagger are identical. Now firstly suppose that $n_l < n_r$. Then (5) is a statement about n_l out of the n_r eigenvalues and eigenvectors of SS^\dagger , and gives no information about the other $n_r - n_l$. Secondly suppose that $n_l > n_r$. As a preparation for the following argument we show that the scalar product of any two different columns of $S(b_{S^\dagger S})$ vanishes. Elements of (4) multiplied by $(a_{S^\dagger S}^*)$ from the left give exactly those scalar products. $(a_{S^\dagger S}^*)$ is the matrix with the complex conjugates of the eigenvectors of $S^\dagger S$ as lines, or $(b_{S^\dagger S})^\dagger$. a_{ij} as before refers to the entries of $(b_{S^\dagger S})$.

$$\begin{aligned} \sum_{i,j,m} S_{ij} a_{jk} (S_{im} a_{ml})^* &= \sum_{i,j,m} a_{ml}^* S_{im}^* S_{ij} a_{jk} = \\ \sum_{i,j,m} a_{ml}^* S_{mi}^\dagger S_{ij} a_{jk} &= \sum_m a_{ml}^* \lambda_k a_{mk} = \delta_{lk} \lambda_k \end{aligned} \quad (6)$$

$S(b_{S^\dagger S})$ has n_l columns, each of which is an n_r -vector. In an n_r -dimensional vector space there can, however, only be n_r non-vanishing mutually orthogonal vectors. To fulfill (6) the remaining $n_l - n_r$ column vectors must be the zero vector. With more modes on the left than the right there are necessarily some modes or linear combinations of modes on the left that do not get transmitted through the junction at all. Now in (6) choose $k = l$ and k one of those column numbers for which $\sum_j S_{ij} a_{jk}$ is zero for all i . Then it follows that the corresponding λ_k , which is an eigenvalue of $S^\dagger S$, is zero as well. For the single junction diagonalizing $S^\dagger S$ or SS^\dagger will lead to the same ensemble of transport channels. For illustration an example with numbers is given in section 4. Regarding the first and last expression in the equation for $l = k$, (6) also tells us that each λ_k represents the sum of some absolute values squared. Therefore by its special construction $S^\dagger S$ not only is a hermitian matrix with real eigenvalues, but all eigenvalues are even greater or equal to zero. This is essential because we shall identify the square roots of the λ_m as phaseless transmission amplitudes t .

3. Double junction

Properties of a series of two junctions will initially be given in terms of two scattering matrices, one between reservoirs L(left) and I(island), and one between reservoirs I and

(a)

$S:$	I_1	I_2	I_3	I_4
L_1	S_{L1I1}	S_{L1I2}	S_{L1I3}	S_{L1I4}
L_2	S_{L2I1}	S_{L2I2}	S_{L2I3}	S_{L2I4}
L_3	S_{L3I1}	S_{L3I2}	S_{L3I3}	S_{L3I4}
R_1	S_{R1I1}	S_{R1I2}	S_{R1I3}	S_{R1I4}
R_2	S_{R2I1}	S_{R2I2}	S_{R2I3}	S_{R2I4}

(b)

$S^\dagger:$	L_1	L_2	L_3	R_1	R_2
I_1	S_{L1I1}^*	S_{L2I1}^*	S_{L3I1}^*	S_{R1I1}^*	S_{R2I1}^*
I_2	S_{L1I2}^*	S_{L2I2}^*	S_{L3I2}^*	S_{R1I2}^*	S_{R2I2}^*
I_3	S_{L1I3}^*	S_{L2I3}^*	S_{L3I3}^*	S_{R1I3}^*	S_{R2I3}^*
I_4	S_{L1I4}^*	S_{L2I4}^*	S_{L3I4}^*	S_{R1I4}^*	S_{R2I4}^*

Figure 2. (a) Scattering matrix and (b) adjoint for the double junction.

R(right) (Fig.3a).§ Multiplying each scattering matrix by its adjoint and diagonalizing would give transport channel ensembles for each junction separately and represent the setup for incoherent coupling [10] across the island. There are multiple reflections in each junction, but a charge transported to the island not to go back through that same junction relaxes there, that is changes the island's electrostatic potential, however, loses the information which (eigen-)mode it had come into. Diagonalizing matrices for each junction separately will not lead to the same linear combinations of island modes as eigenmodes for each side. Setting up our formalism to determine transport channels with coherent coupling between two the junctions is not obvious, should just any of the three reservoirs L,I,R be chosen to demand that eigenmodes there associated to channels exclusively return to themselves if transmitted away from that reservoir and backscattered into it. Nevertheless, as a scattering process starting on the island after two scattering steps offers no other possibility than to have brought the charge carrier back to the island, the above requirement *for the island* provides a promising ansatz.

Therefore given scattering matrices from I to L and from I to R, with by definition the same number of I-modes, are just joint together into one scattering matrix S by taking all rows together. S and S^\dagger are formally given with exemplary numbers of modes three on the left, four on the island and two on the right in Fig.2 (Fig.3 are drawn for different numbers). Now $S^\dagger S$ is diagonalized, again denoting eigenvalues as λ_m and eigenvectors as $(\mathbf{q}_{S^\dagger S})$. The column vectors from $(\mathbf{q}_{S^\dagger S})$ build linear combinations of the original island modes. Eigenvectors of SS^\dagger , however, given by $S(\mathbf{q}_{S^\dagger S})$, now combine modes from both leads L and R. A $\sqrt{\lambda_m}$ associated to an eigenvector out of $(\mathbf{q}_{S^\dagger S})$ or the

§ Still the respective reverse scattering matrices are the complex conjugate transposed ones. With an extended island one could think of a charge being backscattered immediately at the island edge or crossing the island, being backscattered at the rear edge and thus resonator-like interference. For coherent coupling between junctions we are thinking of the island longitudinal dimension between the contacts comparable to the electron coherence length. The transverse dimension may be larger. In contrast to a quantum dot with discrete levels, we want to regard a bulk-like island. The set of original modes on the island should be the same in both scattering matrices, and therefore consist of eigenstates of this bounded space. We thus set the premises for still having a continuous and quasi-constant DOS as a function of energy, however, no extra phase factors to scattering amplitudes complicating the calculation for charges going across the island.

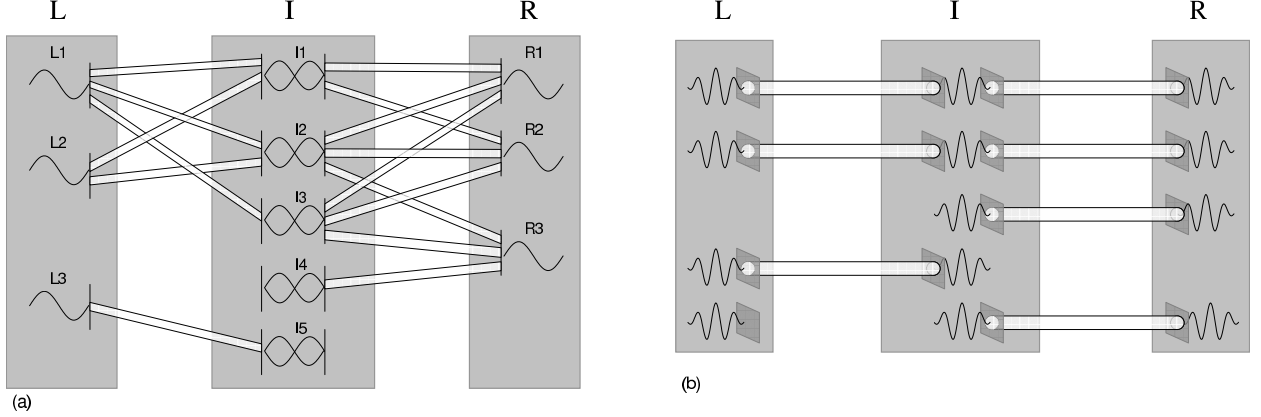


Figure 3. (a) Connections of original modes in a left and right reservoir and on an island between them as given by scattering matrices. (b) Eigenmodes as connected by transport channels.

corresponding vector from $S(k_{S^\dagger S})$ gives us the transmission amplitude of a "combined channel" between the island and the entire outside world consisting of both leads. If there have been n_L original modes at L and n_R at R, every column of $\frac{1}{\sqrt{\lambda_m}} S(k_{S^\dagger S})$ is a normalized vector of length $n_L + n_R$, whose first n_L components represent coefficients for L-modes and whose last n_R components are coefficients for R-modes. However, like for the single junction case, modes at L and R are independent.|| Supplying a charge carrier in an original mode on the left, for example, makes the transmission eigenmodes it contributes to be populated with percentages the absolute values squared of their coefficients in the respective column from $\frac{1}{\sqrt{\lambda_m}} S(k_{S^\dagger S})$. Therefore the transmission of an above mentioned combined channel splits up into the probability percentages left and right side modes contribute to the respective eigenmode. For assigning transmission amplitudes we take the roots of these parts to be multiplied by $\sqrt{\lambda_m}$. So for the transmission amplitudes t_L and t_R for the left and right side channels of a pair of channels coherently linked together on the island (= "combined channel") we obtain

$$t_L = \sqrt{\lambda_m} \sqrt{\sum_{i=1}^{n_L} \left| \frac{1}{\sqrt{\lambda_m}} (S(k_{S^\dagger S}))_{im} \right|^2} \quad (7)$$

and

$$t_R = \sqrt{\lambda_m} \sqrt{\sum_{i=n_L+1}^{n_L+n_R} \left| \frac{1}{\sqrt{\lambda_m}} (S(k_{S^\dagger S}))_{im} \right|^2} \quad (8)$$

This construction demonstrates that a series of point contacts enclosing a bulk-like island even with coherent coupling across the latter is *not* equivalent simply to a number of

|| With the single junction we usually do not regard the mapping of the original modes on a side onto the eigenmodes for transmission any more. For the supply of charge carriers the original modes are considered to be incoherently populated with probability one up the Fermi energy, which automatically makes the eigenmodes be populated each with probability one on each side for they are normalized linear combinations.

effective transport channels from left to right. Paths from left to right rest a channel through the left junction connected to a channel through the right junction and these two parts can have different transmissions. However, we cannot expect to describe coherently coupled junctions by the channel ensembles each one would exhibit as a single junction. Furthermore, by construction there is no coherent cross coupling on the island between our newly found channels. One channel from the left can only be coherently coupled to one channel from the right and vice versa (Fig.3b). This finding elegantly solves the problem of generalizing a Green's functions method for modelling current-voltage characteristics in the coherent case [11] to several channels per junction. The algorithm to calculate changing rates for the island charge is only needed for a single channel per junction. Contributions from all pairs of channels can then simply be added in classical rate equations. There may also be channels in a junction not coherently coupled to a partner channel in the other junction (or pairs with transmission amplitude zero in one half). Their island charging rates are even easier to calculate [10]. We should only expect such unpaired channels, however, if in the original pattern there are modes coupled across one junction that are totally decoupled from all others involved in a network of couplings over both junctions, like L3 and I5 in Fig.3a. I4 is not directly coupled to L, however indirectly via R3 and I3 as well as other paths, and will thus contribute to eigenmodes involving connections to both leads. Like in the single junction case, there may be specific linear combinations of original modes at every one of the three sites that are not transmitted at all.

For a consistent description of the double junction with coherent coupling across the island, it has to be required that $|S^\dagger S \vec{b}| \leq 1$ with the normalized vector \vec{b} representing any linear combination of original modes on the island and S and S^\dagger now the big matrices from Fig.2. The prohibition of probability creation here also implies that all eigenvalues λ_m of $S^\dagger S$ have to be less than or at most one, even if the transmissions assigned to the channels in each junction further get multiplied with the parts by which a new combined island mode is transferred to each side. The new left and right modes at the ends of a pair of channels in fact constitute only parts of new combined-lead modes. The latter, however, form a unique set of eigenmodes. All these considerations about the double junction will be illustrated by number examples in the next section.

We shall here present some further more marginal considerations on probability conservation or prohibiting probability creation. At the beginning of our analysis we regarded the system as described by an orthogonal set of modes on the left, an orthogonal set on the island and an orthogonal set on the right. Of course, with the contact established overlaps exist between modes in a lead and modes on the island. These determine the transition amplitudes put into the original scattering matrix. The overlaps should, however, only go so far that parts of an island wave function that couple to left wave functions rest well separated from parts that couple to the right. Then the sum of absolute squares of all scattering matrix elements out of an island mode will in no case exceed unity. The orthogonality of the modes in one lead ensures that the sum of absolute squares of all couplings to one certain island mode is limited

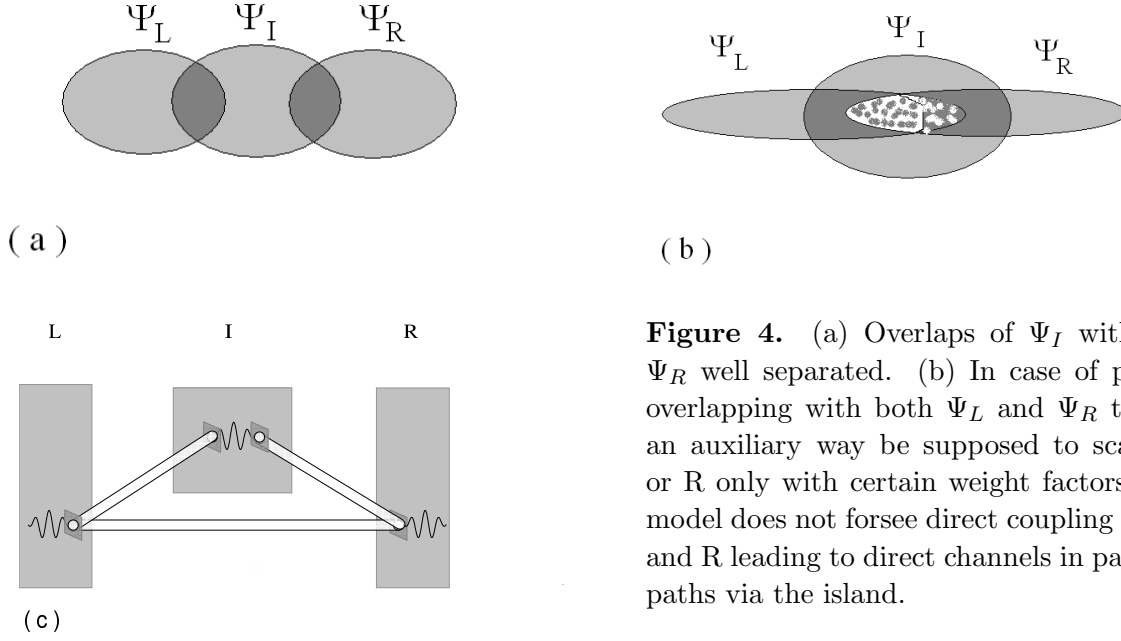


Figure 4. (a) Overlaps of Ψ_I with Ψ_L and Ψ_R well separated. (b) In case of part of Ψ_I overlapping with both Ψ_L and Ψ_R this can in an auxiliary way be supposed to scatter to L or R only with certain weight factors. (c) Our model does not foresee direct coupling between L and R leading to direct channels in parallel with paths via the island.

by the norm one of that mode. Taking just any overlaps that could have been set up as single junction scattering amplitudes does not automatically guarantee that the norm of any column vector of the big matrix S , which contains scattering to the left *and* right modes, is less or equal to one. If the intersection of the overlaps of an island wave function with left wave functions and that with right wave functions is not empty, there would be a direct overlap between left and right wave functions, which will give rise to a direct transport channel from left to right. Such a situation could be realizable, but our model does not cover that. Nevertheless, suppose that two junctions in series have been investigated with incoherent coupling [10]. In the incoherent model changes of the island charge and thus potential are felt by both contacts, but otherwise the latter are independent and behave like a single junction each. With incoherent coupling the regarded set of island modes need not even be the same concerning transport through the left or the right junction, but here suppose it is. To compare with the incoherent behaviour, one might want to estimate the system's properties if the same junctions get linked together coherently (without allowing for direct left-to-right channels) and therefore just put together two single-junction scattering matrices S_{LI} and S_{RI} into one like in Fig.2a. Where this should result in a column norm of S exceeding unity, in this situation it makes sense to renormalize scattering amplitudes in the way that of the island mode the part overlapping with both left and right wave functions is scattered to either side with probabilities proportional to the ratio of absolute squares of the single-junction scattering amplitudes. In any case put-together scattering matrices S should be checked for no column norm exceeding unity prior to solving for transport channels.

4. Examples with numbers

Let us illustrate that the simple criterion of norms of lines or columns being less or equal to 1 is not sufficient to ensure that a matrix is a scattering matrix. For simplicity an example with purely real numbers is given. For the single junction let S^\dagger be

$$S^\dagger = \begin{pmatrix} 0.3 & 0.5 & 0.5 & 0.6 \\ 0.1 & 0.4 & 0.8 & 0.2 \end{pmatrix} \quad (9)$$

supposing two modes on the left and four on the right. Then

$$\begin{aligned} 0.3^2 + 0.5^2 + 0.5^2 + 0.6^2 &= 0.95, 0.1^2 + 0.4^2 + 0.8^2 + 0.2^2 = 0.85, \\ 0.3^2 + 0.1^2 &= 0.10, 0.5^2 + 0.4^2 = 0.41, \\ 0.5^2 + 0.8^2 &= 0.89, 0.6^2 + 0.2^2 = 0.40 \end{aligned} \quad (10)$$

are all less than 1.

$$S^\dagger S = \begin{pmatrix} 0.3 & 0.5 & 0.5 & 0.6 \\ 0.1 & 0.4 & 0.8 & 0.2 \end{pmatrix} \begin{pmatrix} 0.3 & 0.1 \\ 0.5 & 0.4 \\ 0.5 & 0.8 \\ 0.6 & 0.2 \end{pmatrix} = \begin{pmatrix} 0.95 & 0.75 \\ 0.75 & 0.85 \end{pmatrix} \quad (11)$$

The eigenvalues of this matrix, namely the solutions of

$$(0.95 - \lambda)(0.85 - \lambda) - 0.75 \cdot 0.75 = 0, \quad (12)$$

however, are $\lambda_1=0.148$ and $\lambda_2=1.652$ (rounded to 3 digits), of which λ_2 clearly is greater than 1. (3) is not fulfilled:

$$0.95^2 + 0.75^2 + 0.75^2 + 0.85^2 = 2.75 > 1 \quad (13)$$

Let us now regard

$$S = \begin{pmatrix} 0.15 + 0i & 0 - 0.50i \\ 0.10 - 0.15i & 0.10 - 0.05i \\ 0.15 - 0.30i & 0.05 + 0i \end{pmatrix}, \quad (14)$$

which is a scattering matrix, but otherwise its complex entries are randomly chosen. As S is 3x2, there are two original modes on the left and three on the right.

$$S^\dagger S = \begin{pmatrix} 0.1675 & 0.025 - 0.05i \\ 0.025 + 0.05i & 0.265 \end{pmatrix} \quad (15)$$

and the eigenvalues of this 2x2 square matrix are $\lambda_1=0.1421$ and $\lambda_2=0.2904$.

$$SS^\dagger = \begin{pmatrix} 0.2725 & 0.04 - 0.0275i & 0.0225 + 0.02i \\ 0.04 + 0.0275i & 0.045 & 0.065 + 0.005i \\ 0.0225 - 0.02i & 0.065 - 0.005i & 0.115 \end{pmatrix}, \quad (16)$$

is a 3x3 square matrix with eigenvalues $\lambda_1=0.1421$, $\lambda_2=0.2904$ and $\lambda_3=0$ (a numerical calculation gives $2.7 \cdot 10^{-17}$). To check (3) for both $S^\dagger S$ and SS^\dagger we calculate

$$\begin{aligned} &0.1675^2 + 2 \cdot |0.025 - 0.05i|^2 + 0.265^2 = \\ &0.2725^2 + 0.045^2 + 0.115^2 + 2 \cdot |0.04 - 0.0275i|^2 + 2 \cdot |0.0225 + 0.02i|^2 \\ &+ 2 \cdot |0.065 + 0.005i|^2 = 0.105 < 1 \end{aligned} \quad (17)$$

By the way, the same value is obtained as

$$0.2904^2 + 0.1421^2 = 0.105 \quad (18)$$

With complex numbers even normalized eigenvectors are only determined up to a phase common to all components. The numerical routine we used to evaluate them gave them with real last component. For $S^\dagger S$ they are

$$\begin{pmatrix} 0.407 - 0.814i \\ -0.414 + 0i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0.185 - 0.370i \\ 0.910 + 0i \end{pmatrix} \quad (19)$$

associated to λ_1 and λ_2 , respectively. Eigenvectors of SS^\dagger are

$$\begin{pmatrix} -0.085 + 0.123i \\ 0.859 - 0.003i \\ -0.490 + 0i \end{pmatrix}, \quad \begin{pmatrix} 0.277 + 0.020i \\ -0.457 + 0.043i \\ -0.844 + 0i \end{pmatrix}, \quad \begin{pmatrix} 0.881 + 0.354i \\ 0.162 + 0.161i \\ 0.218 + 0i \end{pmatrix} \quad (20)$$

associated to eigenvalues zero, λ_1 and λ_2 in that order. By multiplying (5) by $(\mathbf{a}_{S^\dagger S}^*)S^\dagger$ from the left, one deduces that the factor needed to normalize a column vector of $S(\mathbf{b}_{S^\dagger S})$, which we showed to be an eigenvector of SS^\dagger , is $1/\sqrt{\lambda_m}$ with λ_m the corresponding eigenvalue. We shall now check explicitly the relation between the two sets of eigenvectors in our example, and especially verify that for unequal mode numbers left and right, the transformation in one direction produces some zero vector.

$$\begin{aligned} \frac{1}{\sqrt{0.1421}} \begin{pmatrix} 0.15 + 0i & 0 - 0.5i \\ 0.1 - 0.15i & 0.1 - 0.05i \\ 0.15 - 0.3i & 0.05 + 0i \end{pmatrix} \begin{pmatrix} 0.407 - 0.814i \\ -0.414 + 0i \end{pmatrix} &= e^{0.875i} \begin{pmatrix} 0.277 + 0.02i \\ -0.457 + 0.043i \\ -0.844 + 0i \end{pmatrix} \\ \frac{1}{\sqrt{0.2904}} \begin{pmatrix} 0.15 + 0i & 0 - 0.5i \\ 0.1 - 0.15i & 0.1 - 0.05i \\ 0.15 - 0.3i & 0.05 + 0i \end{pmatrix} \begin{pmatrix} 0.185 - 0.37i \\ 0.91 + 0i \end{pmatrix} &= e^{-1.9i} \begin{pmatrix} 0.881 + 0.354i \\ 0.162 + 0.161i \\ 0.218 \end{pmatrix} \end{aligned} \quad (21)$$

$$\begin{aligned} \frac{\begin{pmatrix} 0.15 + 0i & 0.1 + 0.15i & 0.15 + 0.3i \\ 0 + 0.5i & 0.1 + 0.005i & 0.05 + 0i \end{pmatrix}}{\sqrt{0.1421}} \begin{pmatrix} 0.277 + 0.02i \\ -0.457 + 0.043i \\ -0.844 + 0i \end{pmatrix} &= e^{-0.875i} \begin{pmatrix} 0.407 - 0.814i \\ -0.414 + 0i \end{pmatrix} \\ \frac{\begin{pmatrix} 0.15 + 0i & 0.1 + 0.15i & 0.15 + 0.3i \\ 0 + 0.5i & 0.1 + 0.005i & 0.05 + 0i \end{pmatrix}}{\sqrt{0.2904}} \begin{pmatrix} 0.881 + 0.354i \\ 0.162 + 0.161i \\ 0.218 + 0i \end{pmatrix} &= e^{1.9i} \begin{pmatrix} 0.185 - 0.37i \\ 0.91 + 0i \end{pmatrix} \\ \begin{pmatrix} 0.15 - 0i & 0.1 + 0.15i & 0.15 + 0.3i \\ 0 + 0.5i & 0.1 + 0.005i & 0.05 - 0i \end{pmatrix} \begin{pmatrix} -0.085 + 0.123i \\ 0.859 - 0.003i \\ -0.49 + 0i \end{pmatrix} &= \begin{pmatrix} 1 \cdot 10^{-4} + 0i \\ 5 \cdot 10^{-5} + 1.5 \cdot 10^{-4}i \end{pmatrix} \end{aligned} \quad (22)$$

The last is the zero vector within the accuracy taking into account no more than the written digits. It cannot be normalized, of course, hence there is no factor $1/\sqrt{\lambda_m}$ in the last line.

Entries of scattering matrices are in general complex, but for simplicity we give examples with real numbers for the double junction, which is totally sufficient to show the important aspects of the calculation. To better keep track of which scattering amplitudes belong to the left and the right junction, S is noted in table form like in Fig.2. Our first example has four modes on the island, two in each lead and no vanishing scattering amplitudes.

$S:$	I_1	I_2	I_3	I_4
L_1	0.17	0.28	0.39	0.06
L_2	0.06	0.22	0.40	0.11
R_1	0.28	0.11	0.33	0.39
R_2	0.06	0.33	0.11	0.17

(23)

The sum of squares of all these 16 elements is 0.9921, which is less than 1 and should thus ensure that all eigenvalues will be less than or at most 1. The matrix to diagonalize is

$$S^\dagger S = \begin{pmatrix} 0.1145 & 0.1114 & 0.1893 & 0.1362 \\ 0.1114 & 0.2478 & 0.2698 & 0.1400 \\ 0.1893 & 0.2698 & 0.4331 & 0.2148 \\ 0.1362 & 0.1400 & 0.2148 & 0.1967 \end{pmatrix} \quad (24)$$

As eigenvalues and associated eigenvectors one finds

$$\begin{array}{l} \lambda_m: \quad 0.0080 \quad 0.0518 \quad 0.0959 \quad 0.8364 \\ \\ I_1: \quad \begin{pmatrix} 0.8549 \\ 0.1327 \\ -0.2433 \\ -0.4386 \end{pmatrix} \quad \begin{pmatrix} 0.0728 \\ -0.6389 \\ 0.6469 \\ -0.4100 \end{pmatrix} \quad \begin{pmatrix} -0.3895 \\ 0.5854 \\ 0.1872 \\ -0.6859 \end{pmatrix} \quad \begin{pmatrix} 0.3349 \\ 0.4812 \\ 0.6981 \\ 0.4111 \end{pmatrix} \end{array} \quad (25)$$

S times the matrix made of these four column vectors returns a matrix consisting of the following four column vectors:

$$\begin{array}{l} L_1: \quad \begin{pmatrix} 0.0627 \\ -0.0651 \\ 0.0026 \\ -0.0062 \end{pmatrix} \quad \begin{pmatrix} 0.0612 \\ 0.0775 \\ 0.0039 \\ -0.2050 \end{pmatrix} \quad \begin{pmatrix} 0.1296 \\ 0.1048 \\ -0.2504 \\ 0.0738 \end{pmatrix} \quad \begin{pmatrix} 0.4886 \\ 0.4504 \\ 0.5374 \\ 0.3256 \end{pmatrix} \end{array} \quad (26)$$

(25) gives the normalized eigenvectors of $S^\dagger S$ given in the basis of the original island modes, which is hinted at by the labels I_i in front of their components. (26) lists eigenvectors of SS^\dagger and their components refer to the basis of lead modes. Vectors in (26) are not yet normalized. To do so, divide by the square root of the λ_m above the respective column from (25). To fully link the number examples to the notation of the previous sections, remark that the vectors in (25) give the columns of $(h_{S^\dagger S})$ and those in (26) constitute $S(h_{S^\dagger S})$. Obviously the eigenchannel system consists of four pairs of channels, denoted as (t_{Li}, t_{Ri}) , $i = 1, \dots, 4$. With the values from (26) we shall now work out the

left and right contributions in each. For example, the transmission amplitude of the left channel in the first pair is given by the part of the L-modes in the normalized eigenvector multiplied by the square root of the respective λ_m , which is the transmission amplitude of the island eigenmode to the entire outside world, $t_{L1} = \sqrt{\frac{0.0627^2 + 0.0651^2}{0.0080}} \cdot \sqrt{0.0080}$. The reason to give non-normalized vectors in (26) was that explicit factors $\sqrt{\lambda_m}$ drop out here. The following table lists all left and right transmission amplitudes as determined by (7) and (8):

i	t_{Li}	t_{Ri}
1	$\sqrt{0.0627^2 + 0.0651^2} = 0.0904$	$\sqrt{0.0026^2 + 0.0062^2} = 0.0067$
2	$\sqrt{0.0612^2 + 0.0775^2} = 0.0988$	$\sqrt{0.0039^2 + 0.2050^2} = 0.2050$
3	$\sqrt{0.1296^2 + 0.1048^2} = 0.1667$	$\sqrt{0.2504^2 + 0.0738^2} = 0.2610$
4	$\sqrt{0.4886^2 + 0.4504^2} = 0.6645$	$\sqrt{0.5374^2 + 0.5256^2} = 0.6283$

(27)

Remark that the sum of all t_L squared and divided by the respective λ_m is 2 and the same holds for the t_R .

$$\frac{0.0904^2}{0.0080} + \frac{0.0988^2}{0.0518} + \frac{0.1667^2}{0.0959} + \frac{0.6645^2}{0.8364} = 2.03 \quad (28)$$

$$\frac{0.0067^2}{0.0080} + \frac{0.2050^2}{0.0518} + \frac{0.2610^2}{0.0959} + \frac{0.6283^2}{0.8364} = 1.9992 \quad (29)$$

Deviations from 2 here are due to having limited accuracy to only four digits. (28) and (29) illustrate the reason and purpose of multiplying the $\sqrt{\lambda_m}$ by the weights left and right sides have in the eigenvectors. Despite four channels ending in each lead, the maximum available supply of charges from each lead is reduced to the amount two original modes on each side could provide. We prefer to include these weight factors in t_L and t_R . Alternatively, the $\sqrt{\lambda_m}$ could be called transmission amplitudes of the combined channels. However, then a channel would not connect to equal densities of states on the left and on the right, and weight factors for available states would have to enter effective transfer amplitudes, anyway.

For comparison of the resulting channel transmissions let us now treat the system as incoherent, that is as two single junctions, with the same original scattering amplitudes.

$S_l:$	i_1	i_2	i_3	i_4
l_1	0.17	0.28	0.39	0.06
l_2	0.06	0.22	0.40	0.11

$$S_l S_l^\dagger = \begin{pmatrix} 0.2630 & 0.2344 \\ 0.2344 & 0.2241 \end{pmatrix} \quad (30)$$

$S_l S_l^\dagger$ has eigenvalues 0.0083 and 0.4786, which are also the non-vanishing eigenvalues of $S_l^\dagger S_l$. They correspond to transmission amplitudes $t_{l1} = \sqrt{0.0083} = 0.0911$ and $t_{l2} = \sqrt{0.4786} = 0.6918$.

$S_r:$	i_1	i_2	i_3	i_4
r_1	0.28	0.11	0.33	0.39
r_2	0.06	0.33	0.11	0.17

$$S_r S_r^\dagger = \begin{pmatrix} 0.3515 & 0.1557 \\ 0.1557 & 0.1535 \end{pmatrix} \quad (31)$$

Eigenvalues of $S_r S_r^\dagger$ as well as of $S_r^\dagger S_r$ are 0.0700 and 0.4370 and these translate into transmission amplitudes $t_{r1} = \sqrt{0.0700} = 0.2646$ and $t_{r2} = \sqrt{0.4370} = 0.6610$. Like for the single junction the number of channels in each junction is equal to the smaller number of modes on one side of a junction. There are two channels per junction here. These may, however, have four different linear combinations of original island modes as new modes at their ends on the island. The transmission amplitudes are comparable in size to some of those found in the coherent case, mostly the greater ones for a junction, but there is no easy way of predicting the channel ensembles for the coherent double junction from those for both junctions taken as singles.

In order to verify that modes only coupled across one junction and disconnected from a network extending over both result in channels through only one junction, we have done an analogous calculation to the above one starting with the following scattering matrix:

$S:$	I_1	I_2	I_3	I_4
L_1	0.17	0.28	0.39	0
L_2	0.06	0.22	0.40	0
R_1	0.28	0.11	0.33	0
R_2	0	0	0	0.17

i	t_{Li}	t_{Ri}
1	0.0606	0.0161
2	0	0.17
3	0.1087	0.1497
4	0.6752	0.4204

(32)

This results in three eigenvectors the fourth component of which vanishes (eigenvalues 0.0039, 0.0342 and 0.6326) and one eigenvector of which the first three are zero (eigenvalue 0.0289). The resulting transmission amplitudes of channel pairs are listed in (32). The second line with $t_{L2} = 0$ obviously represents a single channel only bridging the right junction. The input weight of left modes is 2,

$$\frac{0.0606^2}{0.0039} + 0 + \frac{0.1087^2}{0.0342} + \frac{0.6752^2}{0.6326} = 2.01. \quad (33)$$

The input weight of all right modes is also 2,

$$\frac{0.0161^2}{0.0039} + \frac{0.17^2}{0.0289} + \frac{0.1497^2}{0.0342} + \frac{0.4204^2}{0.6326} = 2.001, \quad (34)$$

where 1 falls to the single channel with $t_{R2} = 0.17$ and the others amount to 1 altogether.

The following example is made up for a qualitative estimation on how a junction that exhibits a channel with medium transmission taken as stand-alone device and a tunnel junction that has many low-transmission channels would behave when combined with coherent coupling across the island between them. To make the system as coherent as possible except for paths through the tunnel junction, we assume that many modes from the right lead couple weakly to the same mode on the island. With one mode on the left, one on the island and modes R_1 to R_{10} on the right let the original scattering

amplitudes be given by

$S^\dagger:$	L	R_1	R_2	\dots	R_{10}
I	0.5	0.01	0.01	\dots	0.01

(35)

Then $S^\dagger S = 0.5^2 + 10 \cdot 0.01^2 = 0.251$ and there is only one non-zero $\sqrt{\lambda} = \sqrt{0.251} = 0.501$. The associated eigenvector of $S^\dagger S$ is (I) , that is the island mode is the eigenmode. It translates into the combination $0.5L + 0.01R_1 + 0.01R_2 + \dots + 0.01R_{10}$ in the lead basis. The system exhibits one channel pair with $t_L = \sqrt{0.5^2} = 0.5$ and $t_R = \sqrt{10 \cdot 0.01^2} = 0.032$. From the right only the eigenmode $(0.01R_1 + 0.01R_2 + \dots + 0.01R_{10})/\sqrt{10 \cdot 0.01^2}$ is transmitted, all other linear combinations of right modes are not. The tunnel-junction appears as one effective channel the transmission amplitude of which is enhanced over a single original path by the square root of the number of paths having been assumed with equal throughput. Another example with unequal numbers of original modes left and right will be contained in the next section. There is a tendency of left and right channel transmissions being paired ordered by size. This is no principal necessity, however. For example, the channel left with the largest (smallest) t_L need not be coherently connected to the channel right with the largest (smallest) t_R .

The last example in this section is intended to illustrate how subtle the differences between a valid scattering matrix and a matrix violating probability conservation can be, especially in comparing double to single junctions. Condition (3) is a sufficient criterion, however, set up overcautiously, and thus not a necessary one. We have already seen that if we were to take the matrix from (9) for a single junction between L and I, for example,

$S_l:$	i_1	i_2	i_3	i_4
$l_1:$	0.3	0.5	0.5	0.6
$l_2:$	0.1	0.4	0.8	0.2

(36)

although the norm of each line and each column vector is less than 1, this would result in an eigenvalue and thus a transmission amplitude for one of the two channels $t_2 = \sqrt{\lambda_2} = 1.285$ greater than one. Supposing that the first line gives scattering of four island modes to a single mode on the left, again for only a single junction between L and I,

$S_l:$	i_1	i_2	i_3	i_4
$l:$	0.3	0.5	0.5	0.6

 $S_l S_l^\dagger = (0.95)$ (37)

then it is easily seen from $S_l S_l^\dagger$ that the only non-vanishing eigenvalue is $\lambda = 0.95$ corresponding to $t = \sqrt{0.95} = 0.975$. Analogously supposing that the second line represents scattering through a single junction, named as between island and right,

$S_r:$	i_1	i_2	i_3	i_4
$r:$	0.1	0.4	0.8	0.2

 $S_r S_r^\dagger = (0.85)$ (38)

the eigenvalue and the transmission amplitude for the only channel here would be $\lambda = 0.85$ and $t = 0.922$. If now we were to take the matrix (9) as set up for the double

junction with one mode left and one right

$$\begin{array}{c|c|c|c|c} S: & I_1 & I_2 & I_3 & I_4 \\ \hline L: & 0.3 & 0.5 & 0.5 & 0.6 \\ \hline R: & 0.1 & 0.4 & 0.8 & 0.2 \end{array} \quad \begin{array}{l} \lambda: \quad 0.148 \quad 1.652 \\ \\ L: \quad \begin{pmatrix} 0.6832 \\ -0.7302 \end{pmatrix} \\ R: \quad \begin{pmatrix} -0.7302 \\ -0.6832 \end{pmatrix} \end{array} \quad (39)$$

then from the corresponding eigenvectors of SS^\dagger associated to $\lambda_1=0.148$ and $\lambda_2=1.652$ we get as transmission amplitudes

$$\begin{array}{ccc} i & t_{Li} & t_{Ri} \\ \\ 1 & 0.6832 \cdot \sqrt{0.148} = 0.2628 & 0.7302 \cdot \sqrt{0.148} = 0.2809 \\ 2 & 0.7302 \cdot \sqrt{1.652} = 0.9385 & 0.6832 \cdot \sqrt{1.652} = 0.8781 \end{array}$$

and total mode weights left and right

$$0.2628^2/0.148 + 0.9385^2/1.652 = 1.0 \quad (40)$$

$$0.2809^2/0.148 + 0.8781^2/1.652 = 1.0 \quad (41)$$

Even though a λ exceeds one, no t does. (36) is no allowed scattering matrix for a single junction, but (39) from the result for all t looks possible for a double junction. (We should nevertheless have rejected this S , because (13) tells us that one backscattering off the island of the two lead modes amounts to a weight of more than two, which indicates an overlap situation as in Fig.4b.) These considerations show once more that the double junction has to be treated as a principally different situation and its quantitative properties cannot be deduced in a simple manner from those of the single junctions.

For the algorithms presented here, that is for diagonalizing a matrix $S^\dagger S$ and determining its eigenvalues and even for splitting up such an eigenvalue into amplitudes t_L and t_R for a pair of channels in the double junction, it is of no importance whether the eigenvalues λ are less or equal to one or not. You could be tempted to interpret a $t > 1$ as summing configurationally degenerate modes in a reservoir that quantitatively have identical overlaps with all modes on the other side of a junction and together are well supplied with charge carriers with a weight two, three, etc. instead of one. Such modes should however be entered into S as several, albeit identical, lines or columns to ensure for the orthogonality of all modes and the premises of each being fed with weight one. Transmission amplitudes t greater than one are unphysical. Green's functions [9, 10, 12] that represent transmissions renormalized for multiple reflections require single-hopping amplitudes less or equal to one per transport channel. ¶

Of course, being given scattering matrices for two individual junctions - taking reflection parts with - there is a straight forward rule how to combine two such matrices

¶ The easiest case is $T = t/(1+t^2)$ for a channel in a single junction in the normal conducting case, which stems from a geometric series [13].

into one describing the two junctions put in series (with what we call coherent coupling) [5]. So there has to be a reason why we reopen the problem from the point of view of eigenchannels and newly invent a way for describing the double junction. The fact that for a single junction from experimental results eigenchannel ensembles can be deduced [7], whereas full scattering matrices cannot, does not lead to an irrefutable justification. Merely being given current-voltage characteristics, for the double junction there may even be some ambiguity in deciding whether transport is coherent or incoherent [11]. Like scattering matrices our concept with channel pairs primarily provides a theoretical ansatz for modelling. Convoluting scattering matrices of two junctions in series into one effective matrix once and for all [5] would be appropriate if the island between the junctions did not change its state through charge transport onto and off it (such a resulting matrix would mimic an effective single junction). However, in the situation we are interested in, prone to Coulomb blockade, the island potential changes with each charge transfer and thus the energy range of available electron or hole states does, too. Thus an all-inclusive scattering matrix or a renormalized transfer Green's function has to depend on all possible island charge states. Convoluting here scattering matrices or Green's functions of two junctions into effective global matrices or functions, with the further necessity to transform to Fourier space in order to account for energy-dependent densities of states, is practically impossible for cross-linked transfer paths as in Fig.3a, whereas this is feasible for paired channels as in Fig.3b [12]. Besides that, the original transfer Green's functions approach to transport properties of especially a superconducting quantum point contact [9] was made without separating modes into ingoing and outgoing parts and with restricting scattering matrices to transmission parts (called hopping elements). An extension of that formalism to double junctions should keep these premises.

5. Partial coherence

With today's microstructuring techniques, particularly with metals, it is possible to fabricate islands that are still large enough to exhibit a bulk-like continuous density of states, however already small enough to be sensitive to single-electron charging ⁺ [14]. One may further presume a mixture of coherent and incoherent interaction between processes in both junctions. It shall be shown that such partially coherent transport can be described by finding a system of channels following exactly the same method we just presented for the coherent case. In fact, the system will look like a coherent one with channels of just little lower transmission than one set up with coherent coupling across the island only.

Also the fully coherent case allows sequential (incoherent) transport. A coherent multiscattering process can end on the island letting an electron or hole having come in into any mode there relax into the reservoir of charge carriers equally populating all

⁺ Island dimensions, for example for aluminum, are typically from below one up to two micrometers across and some ten nanometers in height.

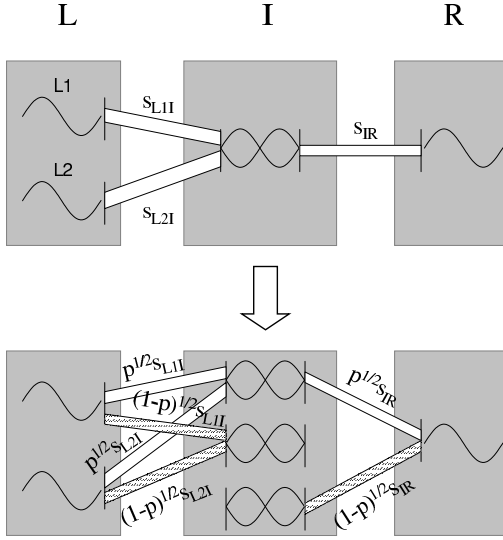


Figure 5. An original island mode with couplings as set up for the fully coherent model is replaced by three new modes coupling to both or only one lead, respectively.

modes. Another process, that discharges the island and brings it back to its original potential, may then begin out of any mode on the island.

To change from full coherence to partial coherence, we want to reduce coherence in transport across the island for a charge carrier coming into a mode there from a lead, but keep the coupling of that island mode to each lead the same, that is not decrease the transmissions of the junctions, an idea reminiscent of [15]. It can here be done by using the following picture. Replace the original island mode which was coupled to modes at L and R by a threefold of new modes: One that is coupled to L- and R-modes as before, but with all scattering amplitudes reduced by a factor of \sqrt{p} , one that is coupled only to the L-modes and one that is coupled only to the R-modes. As scattering amplitudes of the latter two new I-modes take the original amplitudes reduced by a factor of $\sqrt{1-p}$. p is number between 0 and 1 and can be interpreted as the probability that a charge carrier entering into the chosen island mode gets coherently transported across to the other junction. The sum of absolute squares of scattering out of the island mode stays the same as before. For L- or R-modes only their properties localized at a junction count. As compared to the single junction, in contrast to the island in the leads there is no newly introduced specific distance, over which it is important to know to what degree coherence is maintained or lost. For lead modes a splitting as done for island modes does not make sense. Of course, lead and island modes only coupled across one junction and in no way connected to a network coherently linking them to the other lead may always also be present from the beginning.

For comparison with partial coherence we first elaborate the following fully coherent case with two modes on the island, two left and one right.

$S:$	I_1	I_2
L_1	0.17	0.30
L_2	0.06	0.40
R	0.28	0.35

$$S^\dagger S = \begin{pmatrix} 0.1109 & 0.1730 \\ 0.1730 & 0.3725 \end{pmatrix} \quad (42)$$

$S^\dagger S$ has eigenvalues with associated eigenvectors:

$$\begin{array}{cc} 0.0248 & 0.4586 \\ \begin{pmatrix} 0.8953 \\ -0.4455 \end{pmatrix} & \begin{pmatrix} 0.4455 \\ 0.8953 \end{pmatrix} \end{array} \begin{array}{l} \leftarrow I_1 \\ \leftarrow I_2 \end{array} \quad (43)$$

Multiplying by S translates them into eigenvectors of SS^\dagger :

$$\begin{pmatrix} 0.17 & 0.30 \\ 0.06 & 0.40 \\ 0.28 & 0.35 \end{pmatrix} \begin{pmatrix} 0.8953 \\ -0.4455 \end{pmatrix} = \begin{pmatrix} 0.0186 \\ -0.1245 \\ 0.0948 \end{pmatrix} \begin{array}{l} \leftarrow L_1 \\ \leftarrow L_2 \\ \leftarrow R \end{array} \quad (44)$$

$$\begin{pmatrix} 0.17 & 0.30 \\ 0.06 & 0.40 \\ 0.28 & 0.35 \end{pmatrix} \begin{pmatrix} 0.4455 \\ 0.8953 \end{pmatrix} = \begin{pmatrix} 0.3443 \\ 0.3849 \\ 0.4381 \end{pmatrix} \begin{array}{l} \leftarrow L_1 \\ \leftarrow L_2 \\ \leftarrow R \end{array} \quad (45)$$

From left and right parts in these we read off the transmission amplitudes of two pairs of channels:

i	t_{Li}	t_{Ri}
1	$\sqrt{0.0186^2 + 0.1245^2}=0.1259$	0.0948
2	$\sqrt{0.3443^2 + 0.3849^2}=0.5164$	0.4381

In this example remark that the input on the left having a chance to be transmitted amounts to less than 2 modes and that on the right to less than 1.

$$0.1259^2/0.0248 + 0.5164^2/0.4586 = 1.22 \quad (46)$$

$$0.0948^2/0.0248 + 0.4381^2/0.4586 = 0.78 \quad (47)$$

Together they have the weight of 2 modes. The island only offers 2 modes. Thus only a two-dimensional subspace of the three-dimensional vector space spanned by all lead modes can contribute to transmission through the system.

Let us now suppose that the I_2 -mode is split into parts coherently transferred between the junctions and parts losing coherence on the island. The three modes replacing I_2 are I_{2c} for the coherent part and I_{2L} and I_{2R} for the parts coupling to the left and right lead only, respectively. We choose $p = 0.5$. In this special case, $1 - p = 0.5$, too. Compared to column I_2 from S in (42) non-zero numbers in columns I_{2c} , I_{2L} and I_{2R} here are multiplied by $\sqrt{0.5}$, for example, $0.30 \cdot \sqrt{0.5}=0.212$.

$S:$	I_1	I_{2c}	I_{2L}	I_{2R}
L_1	0.17	0.212	0.212	0
L_2	0.06	0.283	0.283	0
R	0.28	0.247	0	0.247

$$S^\dagger S = \begin{pmatrix} 0.1190 & 0.1222 & 0.0530 & 0.0692 \\ 0.1222 & 0.1860 & 0.1250 & 0.0610 \\ 0.0530 & 0.1250 & 0.1250 & 0 \\ 0.0692 & 0.0610 & 0 & 0.0610 \end{pmatrix} \quad (48)$$

Because there is a lead mode less than island modes, one of the eigenvalues, noted below together with the corresponding eigenvectors, is zero.

$$\begin{array}{c}
0 \qquad 0.0059 \qquad 0.1040 \qquad 0.3730 \\
I_1 : \begin{pmatrix} 0 \\ -0.5774 \\ 0.5774 \\ 0.5774 \end{pmatrix} \quad \begin{pmatrix} -0.7583 \\ 0.4085 \\ -0.0912 \\ 0.4997 \end{pmatrix} \quad \begin{pmatrix} -0.4353 \\ 0.0723 \\ 0.6697 \\ -0.5974 \end{pmatrix} \quad \begin{pmatrix} 0.4852 \\ 0.7033 \\ 0.4581 \\ 0.2451 \end{pmatrix}
\end{array} \quad (49)$$

If now we multiply these by S we obtain the (non-normalized) eigenvectors of SS^\dagger , noted again below the corresponding eigenvalues here:

$$\begin{array}{c}
0 \qquad 0.0059 \qquad 0.1040 \qquad 0.3730 \\
L_1 : \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -0.0616 \\ 0.0443 \\ 0.0120 \end{pmatrix} \quad \begin{pmatrix} 0.0833 \\ 0.1839 \\ -0.2516 \end{pmatrix} \quad \begin{pmatrix} 0.3287 \\ 0.3578 \\ 0.3701 \end{pmatrix}
\end{array} \quad (50)$$

The zero-vector here indicates that there is a combination of island modes I_1 , I_{2c} , I_{2L} and I_{2R} , namely that given by the first vector in the above list of four-vectors, which is never transmitted. This state cannot exchange charge carriers with the lead modes. From the other three eigenvectors we read off the transmission amplitudes of three pairs of channels:

i	t_{Li}	t_{Ri}
1	$\sqrt{0.0616^2 + 0.0443^2}=0.0759$	0.0120
2	$\sqrt{0.0833^2 + 0.1839^2}=0.2019$	0.2516
3	$\sqrt{0.3287^2 + 0.3578^2}=0.4859$	0.3701

Here again, the maximum possible input left amounts to two modes, and that right to one:

$$0.0759^2/0.0059 + 0.2019^2/0.1040 + 0.4859^2/0.3730 = 2 \quad (51)$$

$$0.0120^2/0.0059 + 0.2516^2/0.1040 + 0.3701^2/0.3730 = 1 \quad (52)$$

Channels are not at all the same as when letting the I_2 -mode being transported fully coherently across the island. The important finding is, however, that, even if by construction some of the original island modes only have a direct overlap with one lead, the eigenchannel ensemble does not contain channels merely bridging a single junction (if no paths are totally separated from the rest of the original network). The connection

of the coherently and incoherently transported island modes via the lead modes causes the system to appear as effectively consisting of fully coherently linked channel pairs. As already pointed out earlier, the system of channel pairs does not suppress sequential transport. The latter may well be found to provide the dominating contributions in current-voltage characteristics. The channels with the largest throughput left and right may form a pair, in principle, however, as well belong to different pairs ^{*}.

6. Conclusions

It has been discussed how to determine the ensemble of transport channels for a series of two point contacts enclosing an island between them, which allows coherent transport across it. We find that transport channels are only coherently coupled together in pairs involving one channel per junction. This makes a Green's functions algorithm developed for single-channel junctions coherently coupled via an island [11] applicable to the general case of multichannel junctions. The channel ensembles describing coherently coupled contacts will differ from those describing the same contacts each taken as a stand-alone device. Partially coherent transport across the island can also be treated with the presented method of determining transport channels and will effectively look like coherent transport through more but less open channels. Circuits with more than two contacts in series or with short-cutting channels between leads, however, would be more complicated to handle.

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^{*} With coherent coupling, transmission probabilities θ cannot be inferred from single t alone. A renormalization is analytically possible in the normal conducting case, and a channel on the left, for example, would get $\theta_{Li} = 4t_{Li}^2/(1 + t_{Li}^2 + t_{Ri}^2)^2$ [16].

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